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SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

BY ADOLPH VON DER SCHULENBURG.

“IT STILL MOVES!”*

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SINCE I can thus give only a fragment (of my larger work) which is indeed a finished result but finds its complement in the part which must be withheld, I must apologize for external defects. It is to be hoped that, notwithstanding its brevity, it will not be wanting in clearness and intelligibility; and if this should any where still be the case, I can only answer with the apology that the addition of the explanatory operations is not possible under existing circumstances, because of their extent.

On account of this solution of the equation of the 5th degree, much sweat of thought has already flowed. From the old consists who wished to find it to the later analysts who brought system into the attack upon this ‘Veil of Isis,’ it has been an object of algebraic effort; and with comparatively small result. Waring computed the product of differences, Euler knew the right position of the radicals in the roots, Lagrange the solution-sums and Olivier their application to the restoration of the coefficients of a given equation of the 5th degree; but still from the utilization of all these forces in the

*Die Auflösung der Gleichungen funften Grades von Adolf von der Schulenburg. Hauptmann a. D. Halle, bei H. W. Schmidt, 1861.

†My reason for offering this translation for publication is simply the fact that it seems to be unknown among mathematicians in this country that this solution has really been accomplished. As evidence of this generally prevailing impression, I cite the following extract from *Educational Notes and Queries* for February, 1875, Edited by Prof. W. D. Henkle, at Salem, Ohio: “It may not be known to the younger class of mathematicians that the impossibility of resolving algebraically general equations of the fifth or higher degrees has been rigorously demonstrated. This was first done by Abel. Wantzel’s demonstration, which Serret says is simpler than Abel’s, may be seen in Serret’s *Cours D’Algebre Superieure*, 3rd Edition, 1866, Vol. 2.”

Also, the following concluding paragraph from an article by Mr. G. W. Hill in the *ANALYST*, Vol. II, No. 1: “If we should attempt to treat the general equation of the fifth degree in the preceding manner, we would be led to equations of higher degrees than the 5th, which must be regarded as a strong probable argument for the non-existence of an algebraic expression equivalent to the root of the general equation of this degree.”

These statements, as far as I am aware, having passed unchallenged for over a year and a half, I conclude that the solution here given to the American public is not generally known. *Translator.*

solution of the problem, we were so distant that Abel in recent times could bring forward his celebrated proof of the impossibility of this solution: whose abundance of material, whose dialectic keenness, would have fixed a limit to all competent investigation, if the striving after certainty were not so ineradicably planted in the human breast. — *Extract from author's preface.*

Preliminary remark. I use in this memoir two different radical signs, one which admits each root in $\sqrt[n]{a}$, the other which always admits but one of the values in the same formula. When this distinction is not given by the usual symbols and I yet attach a particular weight to it, I remark that here for the most part the reference is to such restricted values. In the few places where this is not the case, it is known by the context.

§. 1.

$$\text{I.} \quad x^5 \pm ax^4 \pm bx^3 \pm cx^2 \pm dx \pm e = 0$$

is an equation of the 5th degree with the coefficients a, b, c, d, e , and the unknown quantity x , whose value and form are to be determined.

After removing any existing coefficient of the term x^5 by dividing the remaining coefficients by it, I. is the most general form of the equation of the 5th degree: complete when a, b, c, d and e have significant values, incomplete when one or more of them $= 0$.

If the roots of I. are taken as positive and rational the complete form becomes

$$\text{II.} \quad x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0,$$

which I retain as general.

By putting one or more of the coefficients a, b, c, d and $e = 0$, several varieties of this form present themselves, from which I particularly select:

$$\text{III. (1)} \quad x^5 + bx^3 - cx^2 + dx - e = 0,$$

the Capital, since it has properties which are of essential importance in the solution of equations generally, and since it especially is the only form characteristic of an exact group which is to be determined;

$$(2) \quad x^5 - e = 0,$$

which form, in restoration of an old notation, I name the Surd-solid.

I do not use any other than these forms.

§. 2.

The equation of the 5th degree has 5 unknown quantities: y, v, z, u, t . If any one, as y , is known, the remaining 4 are found by an equation of the 4th degree.

I denote the five roots of the equation I. $x^5 - 1 = 0$, or the 5 significations of the expression $\sqrt[5]{1}$, by $\alpha(5)'$, $\alpha(5)''$, $\alpha(5)'''$, $\alpha(5)''''$, $\alpha(5)'''''$; I employ the notation $\sqrt[5]{a}$ to denote that, in the formula in which it occurs, any value whatever of $\sqrt[5]{a}$ is signified, but always the same one. Developing this, I find

$$\begin{aligned}\alpha(5)' &= \frac{1}{4}[-1 + \sqrt[5]{5} + \sqrt[5]{-5 + 2\sqrt[5]{5}} + \sqrt[5]{-5 - 2\sqrt[5]{5}}] \text{ or, more briefly,} \\ &= \frac{1}{4}[-1 + \sqrt[5]{5} + p + q] \\ \alpha(5)'' &= \frac{1}{4}[-1 - \sqrt[5]{5} - \sqrt[5]{-5 + 2\sqrt[5]{5}} + \sqrt[5]{-5 - 2\sqrt[5]{5}}] = \frac{1}{4}[-1 - \sqrt[5]{5} - p + q] \\ \alpha(5)''' &= \frac{1}{4}[-1 - \sqrt[5]{5} + \sqrt[5]{-5 + 2\sqrt[5]{5}} - \sqrt[5]{-5 - 2\sqrt[5]{5}}] = \frac{1}{4}[-1 - \sqrt[5]{5} + p - q] \\ \alpha(5)'''' &= \frac{1}{4}[-1 + \sqrt[5]{5} - \sqrt[5]{-5 + 2\sqrt[5]{5}} - \sqrt[5]{-5 - 2\sqrt[5]{5}}] = \frac{1}{4}[-1 + \sqrt[5]{5} - p - q] \\ \alpha(5)''''' &= 1.\end{aligned}$$

These expressions have the property that, if I arbitrarily denote any one of the imaginaries by $\alpha(5)'$, they are as $[\alpha(5)'] : [\alpha(5)']^2 : [\alpha(5)']^3 : [\alpha(5)']^4 : [\alpha(5)']^5$.

The proof is easy: If, e. g., $(\alpha')^2$ should $= n$ and n no one of the remaining imaginary values of $\sqrt[5]{1}$, then I would have $[(\alpha')^2]^5 = n^5 = (\alpha')^5 (\alpha')^5 = 1$. Thus $\sqrt[5]{1} = n$, and $\sqrt[5]{1}$ has more than 5 values.

I designate the combination $y.\alpha(5)'$ briefly by y' .

By considering this and the properties of $\alpha(5)$ already described, I have measurably simplified the operations: e. g., $y'.v''$ can now be written simply yv''' , etc.

§. 3.

By regarding the equation

$$\text{I.} \quad x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$$

as the product of the differences of the general expression for its unknown quantity x and the special values y, v, z, u and t , thus

$$\text{II.} \quad (x - y)(x - v)(x - z)(x - u)(x - t) = 0,$$

I have

$$\text{III.} \quad (y + v + z + u + t) = a,$$

$$\text{IV.} \quad (yv + yz + yu + yt + vz + vu + vt + zu + zt + ut) = b,$$

$$\text{V.} \quad (y vz + y v u + y v t + y z u + y z t + y u t + v z u + v z t + v u t + z u t) = c,$$

$$\text{VI.} \quad (y v z u + y v z t + y v u t + y z u t + v z u t) = d,$$

$$\text{VII.} \quad (y v z u t) = e,$$

from which I am in a position to form next the symmetrical functions of 3, 4, or 5 dimensions.

Having formed in this manner the values of $(y^2 + v^2 + z^2 + u^2 + t^2)$, $(y^3 + v^3 + z^3 + u^3 + t^3)$, $(y^4 + v^4 + z^4 + u^4 + t^4)$, $(y^5 + v^5 + z^5 + u^5 + t^5)$, the sums of the higher powers of y, v, z, u and t are obtained by putting

$$\begin{array}{l} \text{VIII.} \quad y^6 - ay^5 + by^4 - cy^3 + dy^2 - ey = 0 \\ v^6 - av^5 + bv^4 - cv^3 + dv^2 - ev = 0 \\ z^6 - az^5 + bz^4 - cz^3 + dz^2 - ez = 0 \\ u^6 - au^5 + bu^4 - cu^3 + du^2 - eu = 0 \\ t^6 - at^5 + bt^4 - ct^3 + dt^2 - et = 0 \end{array}$$

$$(y^6 + v^6 + z^6 + u^6 + t^6) = a(y^5 + v^5 + z^5 + u^5 + t^5) - b(y^4 + v^4 + z^4 + u^4 + t^4) + c(y^3 + v^3 + z^3 + u^3 + t^3) - d(y^2 + v^2 + z^2 + u^2 + t^2) + ae.$$

Thus I am able to represent every sum of equal powers of the unknown quantities in terms of the coefficients of I., and also, by the help of the powers $(y+v+z+u+t)^n = a^n$, every symmetrical function of the same by means of such coefficients.

I denote a symmetrical function, e. g. that of b , thus: $(yv \dots 10 t)$. In this expression yv signifies the form, and 10 the number of the terms.

§. 4.

The equation of the 5th degree permits the formation of 20 differences of its unknown quantities. The product of these differences is $I. = (108b^5e^2 - 72b^4cde + 16b^4d^3 + 16b^3c^3e - 4b^3c^2d^2 - 900b^3de^2 + 825b^2c^2e^2 + 560b^2cd^2e - 128b^2d^4 - 630bc^3de + 144bc^2d^3 - 3750bce^3 + 2000bd^2e^2 + 108c^5e - 27c^4d^2 + 2250c^2de^3 - 1600cd^3e + 256d^5 + 3125e^4)^*$.

I name ten of these *differences in one direction*, among which there are no two as $+1$:—1, e. g.,

II. $(y-v)(v-u)(z-t)(t-y)(y-z)(z-v)(v-t)(t-u)(u-y)(u-z)$; and the product of *differences in the other direction* is:

III. $(v-y)(u-v)(t-z)(y-t)(z-y)(v-z)(t-v)(u-t)(y-u)(z-u)$.

If expression I. = P , both these products = $\sqrt{P}$.

There are 12 such products of differences in one direction. This product represents a symmetrical function symmetrically divided, viz.:

IV. $(y^4v^3z^2u \dots 120 t)$. While one-half of this function is positive, the other is negative.

From the product of the differences in one direction for an equation of the 5th degree I am able, by assuming one unknown quantity as $t = 0$, to

*Here, as in the equations of the 3d and 4th degrees the innermost square-root of the only irrational quantity occurring in the solution formula always has for its value the product of the differences of the respective unknown quantities, it is assumed that, in the complete representation of the formula for the roots of an equation of the 5th degree, the product of differences also occupies the same position.

construct that of the biquadratic equation, and from this again, those of the cubic and quadratic equations. By doing this I recognize the law which underlies their formation and can now easily write, as by multiplication, the corresponding products of differences for an equation of the 6th degree, etc.

These products of differences in one direction can be again symmetrically analyzed.

§. 5.

I presuppose it known that I can replace a given equation of the 5th degree

I. $x^5 - ax^4 + bx^3 - cx^2 + dx - e = 0$ by its capital:

II. $x_i^5 + b_i x_i^3 - c_i x_i^2 + d_i x_i - e_i = 0.$

I put (one may arbitrarily assume it, on grounds, moreover, which relate to the characteristic of the group of equations whose capital is II.)

III.	$(-2a^2 + 5b)$	$= A$ as the first,
	$(4a^3 - 15ab + 25c)$	$= B$ “ “ second,
	$(-8ac + 3b^2 + 20d)$	$= C$ “ “ third,
	$(8a^2c - 3ab^2 - 50ad + 5bc + 250e)$	$= D$ “ “ fourth

constants of the group, and then for the capital there follows $x_i = x + \frac{1}{5}a$, &

IV. $x_i^5 + \frac{Ax_i^3}{5} - \frac{Bx_i^2}{25} + \frac{(25C - 3A^2)x_i}{500} + \frac{(AB - 25D)}{6250} = 0.$

Therefore I am authorized, henceforth, to operate freely with the capital as a basis, since I can obtain it for any equation, and can refer the result reached, with its application, to the given equation.

The use of the capital affords not only an aid to the calculation, by the elimination of a , but it is essentially necessary for a special purpose, viz., the partition of symmetrical functions.

If I multiply, by way of exercise, the term $yzvu$, a part of $(yzvu + yvzt + yuzt + yvut + vzut) = d$, by $(y+v+z+u+t) = a = 0$, there results

$$y^2vzu + v^2yzu + z^2yvu + u^2y vz + yvzut = 0.$$

Thus: $(y^2vzu + v^2yzu + z^2yvu + u^2y vz) = -e.$

The symmetrical function $(y^2vzu \dots 20t.)$ thus admits of division into 5 parts of equal value $(-e).$

§. 6.

An equation of the 5th degree is solvable if the constants

A or B and D or

A and B and C

each = 0, or if

$$AB = 25D,$$

because then the capital will be replaced by an equation of the 4th degree or the surd-solid. In this manner we can transform every equation into one in which one of the first three constants (but not D) = 0; but since a general solution of an equation of the 5th degree is approached no nearer by this path, I pursue it no further.*

§. 7.

Let there be given: $x^5 + bx^3 - cx^2 + dx - e = 0$.

This equation is solved when I have represented its unknown quantity (y) by a function of b, c, d and e .

And this function can contain only *one* irrational quantity under an *outer* radical of the 5th degree, since otherwise it would have and therefore y , more than a quintuple signification.

Further, upon the ground just stated, the expression for the unknown y , can contain no quantity with an outer radical of higher degree than the 5th.

Whatever is the form of the *dignant*† of the expression $\sqrt[5]{F}$, which may represent this radical of the 5th degree, it may remain provisionally uncertain; but if it contains radical quantities they must occur with exactly determinate values and sustain to one another a common relation in all the formulas, since otherwise again more than five values would be given for y : this value must be relatively rational and susceptible of only one signification.

A similar condition will hold in regard to the coefficients occurring with $\sqrt[5]{F}$ or the powers of this expression.

It is evident, on the other hand, that that one radical quantity can occur in the 1st, 2d, 3d and 4th powers, whereby the chief requisite of the expression, that it yield only five values, is not violated.

By considering this I am able to separate the expression for y into groups, of which one embraces the sum of the terms without the factor $\sqrt[5]{F}$, while four others have, each one, factors which are respectively the 1st, 2d, 3d and 4th powers of $\sqrt[5]{F}$. Thus:

$$\text{II. } y = A + B\sqrt[5]{F} + C(\sqrt[5]{F})^2 + D(\sqrt[5]{F})^3 + E(\sqrt[5]{F})^4,$$

or with briefer notation:

*With the equation of the 3d degree the corresponding process leads to a new mode of solution; and with that of the 4th degree an analogous procedure leads to a solution by a quadratic auxiliary equation.

†As this term is always used with a special signification, I simply *transfer* it from the original.

$$\begin{aligned}y &= A + Bf + Cf^2 + Df^3 + Ef^4 \\v &= A + Bf' + C(f')^2 + D(f')^3 + E(f')^4 \\z &= A + Bf'' + C(f'')^2 + D(f'')^3 + E(f'')^4 \\u &= A + Bf''' + C(f''')^2 + D(f''')^3 + E(f''')^4 \\t &= A + Bf'''' + C(f'''')^2 + D(f'''')^3 + E(f'''')^4;\end{aligned}$$

but since $(f')^2 = f^{2'}$, &c., I write the above thus:

$$\begin{aligned}\text{III.} \quad y &= A + Bf + Cf^2 + Df^3 + Ef^4 \\v &= A + Bf' + Cf^{2'} + Df^{3'''} + Ef^{4'''} \\z &= A + Bf'' + Cf^{2'''} + Df^{3'} + Ef^{4'''} \\u &= A + Bf''' + Cf^{2'} + Df^{3'''} + Ef^{4'''} \\t &= A + Bf'''' + Cf^{2'''} + Df^{3''} + Ef^{4''}\end{aligned}$$

Sum: $(y+v+z+u+t) = 5A = a$, or in this case (capital) = 0.

By now removing $A=0$ and multiplying y, v, z, u and t respectively by $\alpha(5)', \alpha(5)'',$ etc., so that Ef^4 always enters without the imaginary ${}^5\sqrt{1}$, I have:

$$\begin{aligned}\text{IV.} \quad y &= Bf + Cf^2 + Df^3 + Ef^4 \\v' &= Bf'' + Cf^{2'''} + Df^{3''''} + Ef^4 \\z'' &= Bf'''' + Cf^{2'} + Df^{3'''} + Ef^4 \\u''' &= Bf' + Cf^{2''''} + Df^{3''} + Ef^4 \\t'''' &= Bf''' + Cf^{2''} + Df^{3'} + Ef^4 \\ \hline (y + v' + z'' + u''' + t'') &= 5Ef^4; \text{ similarly I find} \\ (y + v'' + z'''' + u' + t''') &= 5Df^3 \\ (y + v''' + z' + u'''' + t'') &= 5Cf^2 \\ (y + v'''' + z'' + u'' + t') &= 5Bf,\end{aligned}$$

whereby it is proved that the roots of an equation of the 5th degree must always be represented as an aggregate of the sums:

$$\text{V.} \quad y = \left\{ \frac{(y+v+z+u+t) + (y+v'+z''+u''' + t'') + (y+v''+z''''+u' + t''') + (y+v''' + z' + u'''' + t'') + (y+v'''' + z'' + u'' + t')}{5} \right\}$$

These sums I call solution-sums. The mutual fitness of any five of these is given by their origin. One of them, the basis of the group, is so chosen that in it one of the unknown quantities is combined with $({}^5\sqrt{1})'$, the second with $({}^5\sqrt{1})''$, the third with $({}^5\sqrt{1})'''$, the fourth with $({}^5\sqrt{1})''''$, and finally the 5th with $({}^5\sqrt{1})''''' = 1$. The four remaining solution-sums belonging to expression V. are now constructed on this basis by combining the 2d, 3d, 4th and 5th powers of the relative ${}^5\sqrt{1}$ with the same unknown quantities.*

*By the 24 possible permutations of v, z, u and t , 23 other groups of such aggregates can

In this case I employ $Bf = (y + v'''' + z''' + u'' + t')$ as the basis. I assume the factor B of the same $= 1$, whereby nothing in the signification of the formula is changed (only the meaning of the individual values f, C, D, E) while the result remains the same, and I write*

$$\begin{aligned}\text{VI.} \quad y &= \frac{1}{5}[f + Cf^2 + Df^3 + Ef^4] \\ v &= \frac{1}{5}[f' + Cf^{2''} + Df^{3'''} + Ef^{4''''}] \\ z &= \frac{1}{5}[f'' + Cf^{2'''} + Df^{3'''} + Ef^{4''''}] \\ u &= \frac{1}{5}[f''' + Cf^{2'''} + Df^{3'''} + Ef^{4'''}] \\ t &= \frac{1}{5}[f'''' + Cf^{2'''} + Df^{3''} + Ef^{4'}]\end{aligned}$$

If from this I deduce the symmetrical functions for b, c, d and e , there results ($f^5 = F$):

$$\begin{aligned}\text{VII.} \quad (1) \quad b &= -\frac{1}{5}[E + CD]F \\ (2) \quad c &= \frac{1}{25}[(CE^2 + D^2E)F^2 + (C^2 + D)F] \\ (3) \quad d &= \frac{1}{125}[-DE^3F^3 + (-C^3E + C^2D^2 - CDE - D^3 + E^2)F^2 - CF] \\ (4) \quad e &= \frac{1}{3125}[(E^5F^4 + D^5F^3 + C^5F^2 + F) + 5(C^2DE^2 - CD^3E \\ &\quad - CE^3 + D^2E^2)F^3 + 5(-C^3D + C^2E + CD^2 - DE)F^2].\end{aligned}$$

Equations VII. (3) and (4) are yet to be abbreviated. From (1) and (2):

$$\begin{aligned}(5) \quad 25b^2 &= (C^2D^2F^2 + E^2F^2 + 2CDEF^2) \\ (6) \quad 125bc &= (C^2DE^2F^3 + CD^3EF^3 + CE^3F^3 + D^2E^2F^3 + C^3DF^2 \\ &\quad + C^2EF^2 + D^2CF^2 + DEF^2) \\ (7) \quad 125d - 25b^2 &= -DE^3F^3 - (C^3E + 3CDE + D^3)F^2 - CF \\ (8) \quad -125bc + 3125e &= (E^5F^4 + D^5F^3 + C^5F^2 + F) + 10(C^2DE^2 \\ &\quad + D^2E^2)F^3 + 10(C^2E + CD^2)F^2.\end{aligned}$$

(To be continued.)

be formed. It is the same whichever one of these is employed in the representation of the formula for y , as in the equation of the third degree

$$\frac{{}_3\sqrt{\frac{B+\sqrt{(B^2-4A^3)}}{2}} + \frac{A}{{}_3\sqrt{\frac{B+\sqrt{(B^2-4A^3)}}{2}}}}{3}$$

represents the unknown quantity y as well as

$$\frac{{}_3\sqrt{\frac{B-\sqrt{(B^2-4A^3)}}{2}} + \frac{A}{{}_3\sqrt{\frac{B-\sqrt{(B^2-4A^3)}}{2}}}}{3}$$

Both are aggregates of solution-sums: in this case certainly the same, since the 3d degree has to employ for the representation of y only the three forms $(y + v + z)$, $(y + va(3))' + za(3)''$ and $(y + va(3)'' + za(3)')$.

If now in V. y is admitted to the possible permutations, 120 solution-sums arise, whose application to the representation of the roots of the 5th degree is given in §. 7 and 8.

*We must now, for example, regard y as written:

$$y = [{}^5\sqrt{(B^5F)}]' + \frac{C}{B}[{}^5\sqrt{(B^5F)}]'' + \frac{D}{B}[{}^5\sqrt{(B^5F)}]''' + \frac{E}{B}[{}^5\sqrt{(B^5F)}]''''.$$